Tutorial #4

Exercise 1

(i)
$$x + y + t = 1$$
$$-z + t = -3$$

Write as a matrix equation:

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$x + y + t = 1$$
$$-z + t = -3$$

$$\begin{cases} x = 1 - y - t \\ z = t + 3 \end{cases}$$

x and z are dependent variables
y and t are free variables.

relabel; let $y = t_1$
$t = t_2$

$$\Rightarrow \vec{x} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 - t_1 - t_2 \\ t_1 \\ t_2 + 3 \\ t_2 \end{pmatrix}$$

$$\vec{x} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}}_{\text{particular solution}} + t_1 \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}} + t_2 \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}}$$

basis vectors of the associated homogeneous system

(ii) $x_4 - x_3 = 1$
$x_3 - x_2 = 2$
$x_2 - x_1 = 3$

Write as a matrix equation

$$\left(\begin{array}{cccc:c} 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 & 2 \\ -1 & 1 & 0 & 0 & 3 \end{array}\right)$$

bring to row echelon form

$$\left(\begin{array}{cccc:c} -1 & 1 & 0 & 0 & 3 \\ 0 & -1 & 1 & 0 & 2 \\ 0 & 0 & -1 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc:c} 1 & -1 & 0 & 0 & -3 \\ 0 & -1 & 1 & 0 & 2 \\ 0 & 0 & -1 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc:c} 1 & -1 & 0 & 0 & -3 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right)$$

$$\left(\begin{array}{cccc:c} 1 & -1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 & -3 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -6 \\ 0 & 1 & 0 & -1 & -3 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right)$$

$$\Rightarrow \quad \begin{aligned} x_1 &= x_4 - 6 \\ x_2 &= x_4 - 3 \\ x_3 &= x_4 - 1 \end{aligned}$$

$\rightarrow$ x_4 = free variable
label as t

$$\begin{aligned} x_1 &= t - 6 \\ x_2 &= t - 3 \\ x_3 &= t - 1 \\ x_4 &= t \end{aligned}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} t-6 \\ t-3 \\ t-1 \\ t \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underbrace{\begin{pmatrix} -6 \\ -3 \\ -1 \\ 0 \end{pmatrix}}_{\text{particular solution}} + t \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{\text{basis of the associated homogeneous system.}}$$

(iii) $x_1 + x_2 + x_3 - x_4 = 3$

$$x_1 = 3 - x_2 - x_3 + x_4$$

let $\left.\begin{aligned} x_2 &= t_1 \\ x_3 &= t_2 \\ x_4 &= t_3 \end{aligned}\right\}$ free variables

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 - t_1 - t_2 + t_3 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

$$\vec{x} = \underbrace{\begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\text{particular solution}} + t_1 \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}} + t_2 \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}} + t_3 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

basis vectors of the associated homogeneous system

Exercise 2

(i) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

→ Find the basis of the row space by finding its reduced row echelon form.

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

⇒ 1^{st} and 2^{nd} rows contain leading 1s so these form a basis for the row space.

$(1, 2, 1) = \vec{r_1}$
$(0, 1, 1) = \vec{r_2}$

$\{\vec{r_1}, \vec{r_2}\}$ is a basis of the row space.

→ the column vectors of A that are at the same positions as the leading 1s of $\tilde{A}$ form a basis of the column space of A.

$$\Rightarrow \{ \vec{c_1}, \vec{c_2} \} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

form a basis of the column space of A.

(ii) $\begin{pmatrix} -3 & -6 \\ 1 & 2 \\ 4 & 8 \end{pmatrix}$

put into row echelon form;

$$\begin{pmatrix} 3 & 6 \\ 1 & 2 \\ 4 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 4 & 8 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow$ the leading 1 is in the 1st row
we can choose $\{(1, 2)\}$ as a basis for the row space of the matrix.

column space

$\rightarrow$ $\vec{c}_1 = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$ $\quad \vec{c}_2 = \begin{pmatrix} -6 \\ 2 \\ 8 \end{pmatrix}$

the column space of the $m \times n$ matrix A is the subspace span $\{\vec{c}_1 \dots \vec{c}_n\} \subseteq \mathbb{R}^m$ spanned by the column vectors of A.

→ The leading 1_s of $\tilde{A}$ are only in the 1^{st} column.

⇒ the basis for the column space is $\{\vec{c_1}\}$

$$\left\{ \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} \right\}$$

[dimension of the column space is 1 → only 1 vector in the basis]

(iii) $\begin{pmatrix} -3 & -6 & 1 \\ 1 & 2 & 1 \\ 4 & 8 & -1 \end{pmatrix}$

→ put into row echelon form;

$$\begin{pmatrix} 1 & 2 & 1 \\ -3 & -6 & 1 \\ 4 & 8 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 4 \\ 4 & 8 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & -5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ the leading 1s are in rows 1 and 2 of $\tilde{A}$

$\Rightarrow$ $(1, 2, 1)$ and $(0, 0, 1)$ form a basis of the row space $\{\vec{r_1}, \vec{r_2}\}$

$\Rightarrow$ basis of the column space;

$$\left\{ \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$